MATH 312

Lecture 8: Freedom

LINEAR INDEPENDENCE AND "BASIS"

Recap from last time: Null & Column Spaces

Q: What is the COMPLETE SOLUTION to

$$A\vec{x} = \vec{b}$$

when $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & -2 & 0 & 0 \\ -1 & 2 & 1 & 2 \\ -2 & 3 & 1 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}$

Ans:

- Look at $[A \mid \vec{b}]$:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 1 \\ 2 & -2 & 0 & 0 & -2 \\ -1 & 2 & 1 & 2 & 2 \\ -2 & 3 & 1 & 2 & 3 \end{array}\right]$$

- Put it in RREF:

$$R = \begin{array}{c} \begin{array}{ccccc} w & x & y & z & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

If $\vec{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$, then: w, x are NOT free (pivots!) but y, z are Free (no pivots)

Get: $\begin{cases} w + y + 2z = 0 \\ x + y + 2z = 1 \end{cases}$ So $\begin{array}{l} w = -y - 2z \\ x = 1 - y - 2z \end{array}$ (All values of y and z are OK!)

- To produce ALL SOLUTIONS of $A\vec{x} = \vec{b}$, use

$$w = -y - 2z$$
$$x = 1 - y - 2z$$

So
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y-2z \\ 1-y-2z \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{constant}} + \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} y}_{y \text{ stuff}} + \underbrace{\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix} z}_{z \text{ stuff}}$$

ANY choices of y and z produce a solution! (This is a 2D space, or a PLANE full of solutions in $\mathbb{R}^4$).

Q What happens when the RREF of $[A \mid \vec{b}]$ is this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

ANS: No solutions! [$\vec{b}$ is NOT in $C(A)$] col space

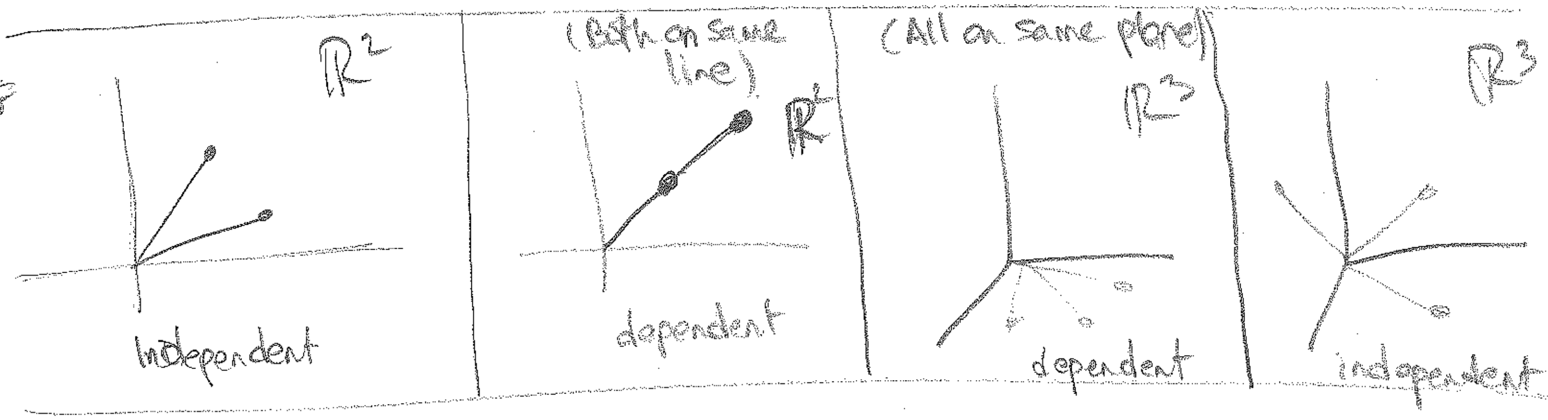
- LINEAR INDEPENDENCE

Def (VERY important) A set of vectors $\vec{v_1}, \dots, \vec{v_n}$ is called $\boxed{\text{LINEARLY INDEPENDENT}}$ if the ONLY linear combination $c_1\vec{v_1} + \dots + c_n\vec{v_n}$ that produces the ZERO vector is given by

$$c_1 = c_2 = \dots = c_n = \underline{0}.$$

- On the other hand, if we find some scalars $c_1, \ldots, c_n$ so that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$ <u>BUT</u> NOT all of the $\vec{c}_j$'s are ZERO, then the vectors $\vec{v}_j$ are said to be <u>LINEARLY DEPENDENT</u>.

Eg



<u>Important</u>: $\vec{v}_1, \ldots, \vec{v}_n$ are linearly Independent

$$\Updownarrow$$

(These are THE SAME).

$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$ only when $c_1 = c_2 = \cdots = c_n = 0$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}_{\text{matrix}} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \text{ only when } c_1 = \cdots = c_n = 0$$

$$\Updownarrow$$

$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ has <u>NULL SPACE</u> $= \{\vec{0} \text{ vector}\}$

$$\Updownarrow$$

The RREF of $[\vec{v}_1, \ldots, \vec{v}_n]$ has NO FREE VARIABLES
(EVERY COLUMNS HAS A PIVOT!)

<u>So</u>: The "easiest" way to check if a collection of vectors is linearly independent or not is to stick 'em in a matrix, compute RREF, and see if every column has a pivot!

If yes, then indep. Otherwise, dep!

So, given (a 2×3 matrix)

$$B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \end{bmatrix},$$

the RREF can't have 3 pivots! At MOST, we might get two: $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$. So, ANY three vectors in $\mathbb{R}^2$ MUST be linearly dependent! And SIMILARLY,

- **if** we have m vectors in $\mathbb{R}^n$
  - and **if** m is STRICTLY larger than n,
    - **then** those vectors are linearly dependent

SPANS:

Let V be your favorite vector space (eg $\mathbb{R}^3$)
Let $\vec{v_1}, \dots, \vec{v_n}$ be a collection of vectors in V
then, we say that $\vec{v_1}, \dots, \vec{v_n}$ SPANS V
if EVERY vector $\vec{b}$ of V is a linear combination of the $\vec{v_1}, \dots, \vec{v_n}$ vectors!

THAT IS, EVERY $\vec{b}$ in V is given by a sum

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{b}$$

THAT IS, Every $\vec{b}$ is in the column space of

$$[\vec{v_1}, \dots, \vec{v_n}]$$

THAT IS No matter WHAT $\vec{b}$ is, the RREF of $[\vec{v}_1, \ldots, \vec{v}_n \mid \vec{b}]$ does **NOT** contain rows of the form

$$\underbrace{0\ 0\ 0 \cdots 0}_{n} \mid 1.$$

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THAT IS, The RREF of $[\vec{v}_1, \ldots, \vec{v}_n]$ has a pivot in **EACH ROW**!

Eg Two vectors in $\mathbb{R}^3$: $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 5 & 4 \end{bmatrix}$ can **NOT** span all of $\mathbb{R}^3$, at best the RREF will have two rows with pivots: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Similarly,

> - if we have m vectors in $\mathbb{R}^n$
> - and if m is STRICTLY smaller than n,
>   - then those vectors can't span $\mathbb{R}^n$

BASIS

Let V be a vector space. Then, the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are a BASIS for V if

- (not too many) they are linearly independent, AND
- (not too few) they span V. (There are infinitely many bases in general!)

> EVERY vector in V is a UNIQUE linear comb of the basis: $\vec{b} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$